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CENTRAL EXTENSIONS OF LOOP GROUPS AND OBSTRUCTION TO PRE-QUANTIZATION

DEREK KREPSKI

ABSTRACT. An explicit construction of a pre-quantum line bundle for the moduli space of flat G-bundles over a Riemann surface is given, where G is any non-simply connected compact simple Lie group. This work helps to explain a curious coincidence previously observed between Toledano-Laredo's work classifying central extensions of loop groups LG and the author's previous work on the obstruction to pre-quantization of the moduli space of flat G-bundles.

1. INTRODUCTION

The moduli space $\mathcal{M}(\Sigma)$ of flat G-bundles over a surface Σ with one boundary component is known to admit a pre-quantization at integer levels[1] when the structure group G is a simply connected compact simple Lie group. If the structure group is not simply connected, however, integrality of the level does not guarantee the existence of a pre-quantization. It was found in [6], that for non-simply connected G, $\mathcal{M}(\Sigma)$ admits a pre-quantization if and only if the underlying level is an integer multiple of $l_0(G)$ listed below in Table 1 for all non-simply connected compact simple Lie groups G.

G	$SU(n)/\mathbb{Z}_k$ $n \geq 2$	$PSp(n)$ $n \geq 1$	$SO(n)$ $n \geq 7$	$PO(2n)$ $n \geq 4$	$Ss(4n)$ $n \geq 2$	PE_6	PE_7
$l_0(G)$	$\mathrm{ord}_k(\frac{n}{k})$	1, n even 2, n odd	1	2, n even 4, n odd	1, n even 2, n odd	3	2

TABLE 1. The integer $l_0(G)$. Notation: $\mathrm{ord}_k(x)$ denotes the order of x mod k in $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

A curiosity observed in [6] is that the integer $l_0(G)$ also appears in Toledano-Laredo's work [10], which classifies positive energy projective representations of loop groups LG for non-simply connected compact simple Lie groups G. To be more specific, Toledano-Laredo classifies central extensions

$$1 \to U(1) \to \widehat{LG} \to LG \to 1,$$

showing they can only exist at levels that are integer multiples of the so-called *basic level* $l_b(G)$, which is then computed for each non-simply connected G (see Proposition 3.5 in [10]). By inspection, it is easy to see that $l_0(G) = l_b(G)$ and this paper aims to understand this coincidence.

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[1]The level $l > 0$ encodes a choice of invariant inner product on the simple Lie algebra $\mathfrak{g}$ of G.

The main result of this work, which helps to account for the observed coincidence, is an explicit construction of a pre-quantum line bundle over the moduli space $\mathcal{M}(\Sigma)$ of flat G-bundles in the case when the structure group G is non-simply connected. The construction is an extension of the well known constructions in the case when G is simply connected (see [9] and [7]). It also appears in [1] for non-simply connected G, although using unnecessary assumptions on the underyling level. The necessary and sufficient condition for pre-quantization, found in [6], is that the underlying level must be an integer multiple of $l_0(G)$. Using the equality $l_0(G) = l_b(G)$, we show that the construction appearing in [1] applies at these levels.

The obstruction to applying this construction of the pre-quantum line bundle in the case of non-simply connected structure group G is related to a central extension

$$1 \to U(1) \to \widehat{\Gamma} \to \Gamma \to 1, \tag{1.1}$$

where $\Gamma \cong \pi_1(G) \times \pi_1(G)$ (see (4.4) in Section 4). The proof of Theorem 4.2 shows that this extension is trivial precisely when the underlying level is an integer multiple of the basic level $l_b(G)$. As a consequence, when the level is an integer multiple of the basic level, the well known construction of the pre-quantum line bundle applies.

This paper is organized as follows. Section 2 reviews some of the relevant background material about loop groups and establishes some notation used throughout the paper. Section 3 reviews the construction of the moduli space, paying special attention to the fact that the underlying structure group is not simply connected. Finally, Section 4 contains the main results of this work, which include a careful study of the central extensions of the gauge groups and Theorem 4.2 whose proof shows that non-triviality of the central extension (1.1) mentioned above is the obstruction to constructing the pre-quantum line bundle. This last section also contains the construction of the pre-quantum line bundle under the conditions when the above central extension is trivial.

2. Preliminaries and Notation

In this section, we establish notation that will be used in the rest of this paper and review some relevant background material.

Let G be a simply connected compact simple Lie group with Lie algebra $\mathfrak{g}$ and let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. For a non-trivial subgroup Z of the center $Z(G)$, let $G' = G/Z$ with maximal torus $T' = T/Z$, which identifies the quotient map $\pi : G \to G'$ as the universal covering homomorphism, and $Z \cong \pi_1(G')$. (Recall that all non-simply connected compact simple Lie groups G' are of this form.)

Let $\Lambda = \ker \exp_T$ be the integer lattice for G and $\Lambda' = \ker \exp_{T'}$ be the integer lattice for G', so that $\Lambda \subset \Lambda'$ and $Z \cong \Lambda'/\Lambda$.

Let $B(-,-)$ denote the *basic inner product*, the invariant inner product on $\mathfrak{g}$ normalized to make short co-roots have length $\sqrt{2}$.

Following [7], throughout this paper we fix a real number $s > 1$. For a given manifold X (possibly with boundary) and $p \leq \dim X$, let $\Omega^p(X;\mathfrak{g})$ be the space of $\mathfrak{g}$-valued p-forms on X of Sobolev class $s - p + \dim X/2$. For a compact Lie group K with Lie algebra $\mathfrak{k}$, the space $\Omega^0(X;\mathfrak{k}) = \mathrm{Map}(X, \mathfrak{k})$ is the Lie algebra of the group $\mathrm{Map}(X, K)$ of maps of Soboloev class $s + \dim X/2$.

Loop groups and central extensions. For a compact Lie group K with Lie algebra $\mathfrak{k}$, let LK denote the (free) loop space $\mathrm{Map}(S^1, K)$, viewed as an infinite dimensional Lie group, with Lie algebra $L\mathfrak{k} = \mathrm{Map}(S^1, \mathfrak{k})$.

Given an invariant inner product $(-,-)$ on $\mathfrak{k}$, define the central extension $\widehat{L\mathfrak{k}} := L\mathfrak{k} \oplus \mathbb{R}$ with Lie bracket

$$[(\xi_1, t_1), (\xi_2, t_2)] := ([\xi_1, \xi_2], \int_{S^1} (\xi_1, d\xi_2)).$$

If it exists, let $\widehat{LK}$ denote a $U(1)$-central extension of LK with Lie algebra $\widehat{L\mathfrak{k}}$.

For $K = G$, it is well known (see Theorem 4.4.1 in [8]) that central extensions $\widehat{LG}$ are classified by their *level* l—the unique multiple of the basic inner product that coincides with the chosen inner product—which is required to be a positive integer. (Since G is simple, any invariant inner product on $\mathfrak{g}$ is necessarily of the form $lB(-,-)$ for some $l > 0$ called the *level*.)

For $K = G'$, however, central extensions $\widehat{LG'}$ are classified by their level l, which is required to be an integer multiple of $l_b(G')$, and a character $\chi : Z \to U(1)$ (see Proposition 3.4 in [10]). The integer $l_b(G')$ is defined as follows.

Definition 2.1. Let G' be a compact simple Lie group with integer lattice Λ'. The *basic level* $l_b(G')$ is the smallest integer l such that the restriction of $lB(-,-)$ to Λ' is integral.

As mentioned in the introduction, $l_b(G') = l_0(G')$, which appears in Table 1 for each non-simply connected compact simple Lie group G'.

Let $L\mathfrak{g}^* = \Omega^1(S^1; \mathfrak{g})$, sometimes called the *smooth dual* of $L\mathfrak{g}$. The pairing $L\mathfrak{g} \times L\mathfrak{g}^* \to \mathbb{R}$ given by $(\xi, A) \mapsto \int_{S^1}(\xi, A)$, induces an inclusion $L\mathfrak{g}^* \subset (L\mathfrak{g})^*$. Additionally, define $\widehat{L\mathfrak{g}}^* := L\mathfrak{g}^* \oplus \mathbb{R}$ and consider the pairing $\widehat{L\mathfrak{g}} \times \widehat{L\mathfrak{g}}^* \to \mathbb{R}$ given by

$$((\xi, a), (A, t)) = \int_{S^1} (\xi, A) + at.$$

Since the central subgroup $U(1) \subset \widehat{LG}$ acts trivially on $\widehat{L\mathfrak{g}}$, the coadjoint representation of $\widehat{LG}$ factors through LG. The coadjoint action of LG on $\widehat{L\mathfrak{g}}^*$ is (see Proposition 4.3.3 in [8]):

$$g \cdot (A, t) = (\mathrm{Ad}_g(A) - tg^*\theta^R, t)$$

where θ^R denotes the right-invariant Maurer-Cartan form on G.

Notice that for each real number λ, the hyperplanes $t = \lambda$ are fixed. Identifying $L\mathfrak{g}^*$ with $L\mathfrak{g}^* \times \{\lambda\} \subset \widehat{L\mathfrak{g}}^*$ yields an action of LG on $L\mathfrak{g}^*$, called the (affine) level λ action.

3. The moduli space of flat connections $\mathcal{M}'(\Sigma)$

In this section, we review the construction of the moduli space of flat connections following [1], with special attention to the case where G' is a non-simply connected compact simple Lie group. The reader may wish to consult [1, 2, 7] and the references therein for more details.

Let Σ denote a compact, oriented surface of genus h with 1 boundary component. The affine space of connections $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ on the trivial G'-bundle over

Σ admits an action of $\mathrm{Map}(\Sigma, G')$, the space of maps $g : \Sigma \to G'$, by gauge transformations

$$g \cdot A = \mathrm{Ad}_g A - g^*\theta^R.$$

The kernel of the restriction map

$$\mathrm{Map}(\Sigma, G') \to \mathrm{Map}(\partial\Sigma, G'), \qquad g \mapsto g|_{\partial\Sigma}$$

will be denoted $\mathrm{Map}_\partial(\Sigma, G')$. Define the moduli space of flat G'-connections up gauge transformations whose restriction to $\partial\Sigma$ is trivial by

$$\mathcal{M}'(\Sigma) := \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathrm{Map}_\partial(\Sigma, G').$$

The Atiyah-Bott [2] symplectic structure on $\mathcal{M}'(\Sigma)$ is obtained by symplectic reduction (as in Chapter 23 of [4]), viewing the moduli space as a symplectic quotient of the affine space of connections $\mathcal{A}(\Sigma)$. Recall that the affine space $\mathcal{A}(\Sigma)$ carries a symplectic form $\omega_{\mathcal{A}}(a_1, a_2) = \int_\Sigma lB(a_1, a_2)$ and a Hamiltonian action of $\mathrm{Map}_\partial(\Sigma, G')$ with momentum map the curvature; therefore, the zero level set of the moment map is the space of flat connections $\mathcal{A}_{\mathrm{flat}}(\Sigma)$ and hence the resulting symplectic quotient is the moduli space $\mathcal{M}'(\Sigma)$.

The moduli space $\mathcal{M}'(\Sigma)$ carries an action by LG which can be described as follows. For $g \in \mathrm{Map}(\Sigma, G')$, the restriction $g|_{\partial\Sigma}$ is a contractible loop in G', since $\pi_1(G')$ is Abelian and $\partial\Sigma$ is homotopic to a product of commutators $\prod a_i b_i a_i^{-1} b_i^{-1}$ for loops a_i, b_i representing generators of $\pi_1(\Sigma)$. Thus the restriction map takes values in the identity component $\mathrm{Map}_0(\partial\Sigma, G')$, which after choosing a parametrization $\partial\Sigma \cong S^1$ can be identified with the identity component L_0G' of the loop group LG'. The LG action on $\mathcal{M}'(\Sigma)$ is then defined using the natural projection $L\pi : LG \to LG'$, which takes values in L_0G', and the identification $\mathrm{Map}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \cong L_0G'$. The LG action is Hamiltonian, with momentum map $\Phi' : \mathcal{M}'(\Sigma) \to L\mathfrak{g}^*$ given by pulling back the connection to the boundary.

The corresponding moduli space $\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}/\mathrm{Map}_\partial(\Sigma, G)$ with simply connected structure group G is a finite covering of $\mathcal{M}'(\Sigma)$. This is a consequence of the following Proposition found in [1].

Proposition 3.1. *The following sequences are exact.*

$$1 \to Z \to \mathrm{Map}(\Sigma, G) \to \mathrm{Map}(\Sigma, G') \to Z^{2h} \to 1 \tag{3.1}$$

$$1 \to \mathrm{Map}_\partial(\Sigma, G) \to \mathrm{Map}_\partial(\Sigma, G') \to Z^{2h} \to 1 \tag{3.2}$$

In the sequences (3.1) and (3.2), the maps into Z^{2h} are defined by sending $g \mapsto g_\sharp$ in $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G')) \cong Z^{2h}$. Since $A \in \mathcal{A}(\Sigma)$ may be viewed as either a G-connection or a G'-connection on the corresponding trivial bundle over Σ, the moduli space $\mathcal{M}(\Sigma)$ admits a residual $Z^{2h} \cong \mathrm{Map}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ action, identifying $\mathcal{M}'(\Sigma) = \mathcal{M}(\Sigma)/Z^{2h}$. Also, the momentum map $\Phi : \mathcal{M}(\Sigma) \to L\mathfrak{g}^*$ is clearly invariant under the Z^{2h}-action and descends to the momentum map $\Phi' : \mathcal{M}'(\Sigma) \to L\mathfrak{g}^*$ above. Viewed this way, Φ' sends an equivalence class of G'-connections to its restriction to the boundary, considered as a G-connection on $\partial\Sigma$.

For $\mu \in L\mathfrak{g}^*$, the symplectic quotient

$$\mathcal{M}(\Sigma)_\mu := \Phi^{-1}(LG \cdot \mu)/LG$$

represents the moduli space of flat connections on the trivial G bundle over Σ whose restriction to the boundary is gauge equivalent to μ. Equivalently, $\mathcal{M}(\Sigma)_\mu$

is the moduli space of flat connections on the trivial G-bundle whose holonomy along the boundary is conjugate to $\mathrm{Hol}(\mu)$. Similarly, the symplectic quotient $\mathcal{M}'(\Sigma)_\mu = (\Phi')^{-1}(LG \cdot \mu)/LG$ represents the moduli space of flat connections on the trivial G'-bundle over Σ whose holonomy along the boundary, when viewed as a G-connection on $\partial\Sigma$, is conjugate to $\mathrm{Hol}(\mu)$.

The connected components of the moduli space of flat G'-bundles over a *closed* surface may then be described in terms of the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$ with $\mathrm{Hol}(\mu) \in Z$. To see this, let $\hat{\Sigma}$ be the closed surface obtained by gluing a disc D to Σ by identifying boundaries. Recall that there is a bijective correspondence between isomorphism classes of principal G'-bundles $P \to \hat{\Sigma}$ and $\pi_1(G') \cong Z$: every such bundle $P \to \hat{\Sigma}$ is isomorphic to one that can be constructed by gluing together trivial bundles over both Σ and D with some transition function $f : S^1 = \Sigma \cap D \to G'$. By Proposition 4.33 in [3], the holonomy around $\partial\Sigma$ of a flat connection on P coincides with $[f] \in \pi_1(G') \cong Z$. It follows that the moduli space $M_{G'}(\hat{\Sigma})$ of flat G'-bundles over a closed surface $\hat{\Sigma}$ up to gauge transformations may be written as the (disjoint) union of the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$ where $\mathrm{Hol}(\mu) \in Z$.

4. The pre-quantum line bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$

In this section, we construct a pre-quantum line bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$, which is an adaptation of a well known construction in the case where the underlying structure group is simply connected (see [9] and [7]). The construction appears in [1], however using unnecessary assumptions on the underlying level. The main contribution here is to verify that this construction applies under the necessary and sufficient conditions obtained in [6]. For simplicity, we consider the case of genus $h = 1$.

Central extensions of the gauge group. An important part of the construction of the pre-quantum line bundle is a careful discussion of certain central extensions of various gauge groups.

Recall that the cocycle defined by the formula $c(g_1, g_2) = \exp i\pi \int_\Sigma lB(g_1^*\theta^L, g_2^*\theta^R)$ defines central extensions

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G) \to \mathrm{Map}(\Sigma, G) \to 1,$$

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G') \to \mathrm{Map}(\Sigma, G') \to 1. \tag{4.1}$$

It is known (see p. 431 in [7]) that when l is an integer, the restriction of the central extension $\widehat{\mathrm{Map}}(\Sigma, G)$ to the subgroup $\mathrm{Map}_\partial(\Sigma, G)$ is trivial; that is the exact sequence

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G) \to \mathrm{Map}_\partial(\Sigma, G) \to 1 \tag{4.2}$$

splits and we may view $\mathrm{Map}_\partial(\Sigma, G) \subset \widehat{\mathrm{Map}}(\Sigma, G)$.

More precisely, the section $\sigma : \mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G)$, $g \mapsto (g, \alpha(g))$ composed with the inclusion $\widehat{\mathrm{Map}}_\partial(\Sigma, G) \hookrightarrow \widehat{\mathrm{Map}}(\Sigma, G)$ embeds $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup in $\widehat{\mathrm{Map}}(\Sigma, G)$, where $\alpha : \mathrm{Map}_\partial(\Sigma, G) \to U(1)$ is defined as follows. For $g \in \mathrm{Map}_\partial(\Sigma, G)$, choose a homotopy $H : \Sigma \times [0, 1] \to G$ with $H_0 = g$, $H_1 = e$ and $H_t|_{\partial\Sigma} = e$ for $0 \le t \le 1$ and define

$$\alpha(g) = \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma \times [0,1]} H^*\eta,$$

where $\eta = B(\theta^L, [\theta^L\theta^L])$ denotes the canonical invariant 3-form on G. It is straightforward to check that α is well-defined and satisfies the coboundary relation

$$\alpha(g_1g_2) = \alpha(g_1)\alpha(g_2)c(g_1, g_2)$$

so that σ is indeed a section. That we may view $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup of $\widehat{\mathrm{Map}}(\Sigma, G)$ is also straightforward (*cf.* Lemma 4.1 and the proof of Corollary 4.3 below).

Therefore, one obtains the central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G)/\mathrm{Map}_\partial(\Sigma, G) \to LG \to 1$$

using the identification $LG \cong \mathrm{Map}(\Sigma, G)/\mathrm{Map}_\partial(\Sigma, G)$.

Assume that l is an integer. Under additional restrictions on l, described in Theorem 4.2, the same holds for the central extension $\widehat{\mathrm{Map}}(\Sigma, G')$ in (4.1) and we obtain a central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \to L_0G' \to 1$$

using the identification $L_0G' \cong \mathrm{Map}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G')$.

Lemma 4.1. *Let* $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ *denote the restriction of the central extension (4.1) to* $\mathrm{Map}_\partial(\Sigma, G')$. *Using the section* $\sigma : \mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G)$ *above and the inclusion* $\widehat{\mathrm{Map}}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ *induced from the inclusion in (3.2), we may embed* $\mathrm{Map}_\partial(\Sigma, G)$ *as a normal subgroup in* $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$.

Proof. The inclusion $\mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ is given by $g \mapsto (\pi g, \alpha(g))$, where $\pi : G \to G'$ is the universal covering homomorphism. To verify that this includes $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup, a direct calculation shows that it suffices to verify that for $g \in \mathrm{Map}_\partial(\Sigma, G)$ and $h \in \mathrm{Map}_\partial(\Sigma, G')$. (Note that $c(h, h^{-1}) = 1$, since $(h^*\theta^L, (h^{-1})^*\theta^R) = -h^*(\theta^L, \theta^L) = 0$.)

$$\alpha(h\pi gh^{-1}) = c(h, \pi gh^{-1})c(\pi g, h^{-1})\alpha(g). \tag{4.3}$$

Note that $h\pi gh^{-1}$ is clearly in $\mathrm{Map}_\partial(\Sigma, G)$ (using the inclusion of (3.2)) so that $\alpha(h\pi gh^{-1})$ is defined.

To compute $\alpha(h\pi gh^{-1})$, let $F : \Sigma \times [0, 1] \to G$ be a homotopy for g such that $F_0 = g$, $F_1 = e$ and $F_t|_{\partial\Sigma} = e$ and let $H : \Sigma \times [0, 1] \to G'$ be the homotopy $H(p, t) = h(p)\pi F(p, t)h(p)^{-1}$. Since $\pi : G \to G'$ is a covering projection, we may lift H to a homotopy $\tilde{H} : \Sigma \times [0, 1] \to G$, and find that

$$\begin{aligned} \alpha(h\pi gh^{-1}) &= \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma\times[0,1]} \tilde{H}^*\eta \\ &= \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma\times[0,1]} (h\pi Fh^{-1})^*\eta. \end{aligned}$$

A direct calculation now verifies that equation (4.3) holds. (See the proof of Corollary 4.3 for a sketch of a similar calculation.) □

Theorem 4.2. *The restriction of the central extension (4.1) to* $\mathrm{Map}_\partial(\Sigma, G')$ *splits if the underlying level* l *is a multiple of the basic level* $l_b(G')$.

Proof. It will be useful in what follows to choose representative loops in $T' \subset G'$ for elements of $Z \cong \pi_1(G')$. For each $z \in Z \cong \Lambda'/\Lambda$ let $\zeta_z \in \Lambda'$ be a (minimal dominant co-weight) representative for z. In particular, $\exp \zeta_z = z \in T \subset G$ and the loop $\zeta_z(t) = \exp(t\zeta_z)$ in $T' \subset G'$ represents z viewed as an element of $\pi_1(G')$.

For $\mathbf{z} = (z_1, z_2) \in Z \times Z$, construct a map $g_{\mathbf{z}} : \Sigma \to G'$ in $\mathrm{Map}_\partial(\Sigma, G')$ as follows. View the surface Σ as the quotient of the pentagon with oriented sides identified according to the word $aba^{-1}b^{-1}c$. Define $g : S^1 \to T'$ on the boundary of the pentagon so that $g|_a = \zeta_{z_1}$, $g|_b = \zeta_{z_2}$ and $g|_c = 1$. Since $\pi_1(T)$ is abelian, g is null homotopic and can be extended to the pentagon, defining $g_{\mathbf{z}} : \Sigma \to T' \to G'$. Note that the induced map $(g_{\mathbf{z}})_\sharp : \pi_1(\Sigma) \to \pi_1(G')$ satisfies $(g_{\mathbf{z}})_\sharp(a) = z_1$ and $(g_{\mathbf{z}})_\sharp(b) = z_2$ and hence $(g_{\mathbf{z}})_\sharp = \mathbf{z}$ in the sequence (3.2).

Since the sequence (4.2) splits, and by Lemma 4.1 we may view $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup of $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$, the restriction of the central extension (4.1) to $\mathrm{Map}_\partial(\Sigma, G')$. Hence, by the exact sequence (3.2), we obtain a central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G) \to Z \times Z \to 1. \tag{4.4}$$

Therefore, the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ fits in the following pullback diagram

$$\begin{tikzcd}
\widehat{\mathrm{Map}}_\partial(\Sigma, G') \arrow[r] \arrow[d] & \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G) \arrow[d] \\
\mathrm{Map}_\partial(\Sigma, G') \arrow[r] & Z \times Z
\end{tikzcd}$$

where the map on the bottom of the square is the one appearing in (3.2). It follows that the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ splits if the central extension (4.4) is trivial.

Central $U(1)$-extensions over the abelian group $\Gamma = Z \times Z$ are determined by their commutator pairing $q : \Gamma \times \Gamma \to U(1)$. (In general, a trivial commutator pairing would only show that the given extension is abelian. However, abelian $U(1)$-extensions are necessarily trivial since $U(1)$ is divisible.) For $\mathbf{z}$ and $\mathbf{w}$ in $Z \times Z$, recall that the commutator pairing is defined by

$$q(\mathbf{z}, \mathbf{w}) = \hat{\mathbf{z}}\hat{\mathbf{w}}\hat{\mathbf{z}}^{-1}\hat{\mathbf{w}}^{-1}$$

where $\hat{\mathbf{z}}$ and $\hat{\mathbf{w}}$ in $\widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ are arbitrary lifts of $\mathbf{z}$ and $\mathbf{w}$ respectively.

Next, we compute the commutator pairing q and determine when it is trivial. To that end, let $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ be constructed as above. Then since $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ lie in T', $g_{\mathbf{z}} g_{\mathbf{w}} = g_{\mathbf{w}} g_{\mathbf{z}}$, and

$$(g_{\mathbf{z}}, 1)(g_{\mathbf{w}}, 1)(g_{\mathbf{z}}, 1)^{-1}(g_{\mathbf{w}}, 1)^{-1} = (1, c(g_{\mathbf{z}}, g_{\mathbf{w}})c(g_{\mathbf{w}}, g_{\mathbf{z}})^{-1}).$$

Therefore,

$$\begin{aligned}
q(\mathbf{z}, \mathbf{w}) &= c(g_{\mathbf{z}}, g_{\mathbf{w}})c(g_{\mathbf{w}}, g_{\mathbf{z}})^{-1} \\
&= \exp \pi i \int_\Sigma (lB(g_{\mathbf{z}}^*\theta^L, g_{\mathbf{w}}^*\theta^R) - lB(g_{\mathbf{w}}^*\theta^L, g_{\mathbf{z}}^*\theta^R)) \\
&= \exp 2\pi i \int_\Sigma lB(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta)
\end{aligned}$$

where θ denotes the Maurer-Cartan form on the torus T'.

By collapsing the boundary of Σ to a point, we map view the maps $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ as maps from the 2-torus $T^2 \to T'$. If ω denotes the standard symplectic form on T^2 with unit symplectic volume, then $lB(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta) = (lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{z_2}, \zeta_{w_1}))\omega$. Indeed,

$$\begin{aligned}(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta)((u_1,u_2),(v_1,v_2)) &= lB(\theta(g_{\mathbf{z}*}(u_1,u_2)), \theta(g_{\mathbf{w}*}(v_1,v_2))) \\ &\quad - lB(\theta(g_{\mathbf{z}*}(v_1,v_2)), \theta(g_{\mathbf{w}*}(u_1,u_2))) \\ &= lB(u_1\zeta_{z_1} + u_2\zeta_{z_2}, v_1\zeta_{w_1} + v_2\zeta_{w_2}) \\ &\quad - lB(v_1\zeta_{z_1} + v_2\zeta_{z_2}, u_1\zeta_{w_1} + u_2\zeta_{w_2}) \\ &= (lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{z_2}, \zeta_{w_1}))(u_1v_2 - v_1u_2)\end{aligned}$$

Therefore,

$$q(\mathbf{z}, \mathbf{w}) = \exp 2\pi i \,(lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{w_1}, \zeta_{z_2}))$$

and q is trivial if and only if l is a multiple of the basic level $l_b(G')$. □

Corollary 4.3. *If the level is an integer multiple of the basic level, there is a central extension*

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \to L_0G'.$$

Proof. As in the proof of Theorem 4.2, at any integer level, the central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G') \to \mathrm{Map}_\partial(\Sigma, G') \to 1$$

is the pullback of the central extension (4.4) over the abelian group $Z \times Z$. Moreover, if the underlying level is a multiple of the basic level, the proof of Theorem 4.2 shows that this extension is abelian and hence split.

Each choice of section $\delta : Z \times Z \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ of the central extension (4.4) induces a canonical section $s : \mathrm{Map}_\partial(\Sigma, G') \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ as follows. For $g \in \mathrm{Map}_\partial(\Sigma, G')$, write $\delta(g_\sharp) = [(h, z)]$. Since $h_\sharp = g_\sharp$, by the exactness of (3.2), there is a unique $a \in \mathrm{Map}_\partial(\Sigma, G)$ with $h\pi a = g$. Define

$$s(g) = (g, c(h, \pi a)z\alpha(a)).$$

It is easy to check that s is well-defined and is indeed a section. It remains to verify that the induced inclusion $\mathrm{Map}_\partial(\Sigma, G') \xrightarrow{s} \widehat{\mathrm{Map}}_\partial(\Sigma, G') \hookrightarrow \widehat{\mathrm{Map}}(\Sigma, G')$ includes $\mathrm{Map}_\partial(\Sigma, G')$ as a normal subgroup.

To that end, observe first that it suffices to check that $\mathrm{Map}_\partial(\Sigma, G')$ is closed under conjugation by elements of $\widehat{\mathrm{Map}}(\Sigma, G')$ in the image of $\widehat{\mathrm{Map}}(\Sigma, G) \to \widehat{\mathrm{Map}}(\Sigma, G')$ induced from (3.1). Indeed, the sequences (3.1) and (3.2) show that each k in $\mathrm{Map}(\Sigma, G')$ can be expressed as $k = \pi x f$ where $f \in \mathrm{Map}_\partial(\Sigma, G')$ satisfies $k_\sharp = f_\sharp$ and $x \in \mathrm{Map}(\Sigma, G)$.

Let $g \in \mathrm{Map}_\partial(\Sigma, G')$ and choose $x \in \mathrm{Map}(\Sigma, G)$. Then

$$(\pi x, w)s(g)(\pi x, w)^{-1} = (\pi x g \pi x^{-1}, c(\pi x g, \pi x^{-1})c(\pi x, g)c(h, \pi a)z\alpha(a))$$

where $\delta(g_\sharp) = [(h, z)]$ and $h\pi a = g$ for $a \in \mathrm{Map}_\partial(\Sigma, G)$. Since $(\pi x g \pi x^{-1})_\sharp = g_\sharp$, then $s(\pi x g \pi x^{-1}) = (\pi x g \pi x^{-1}, c(h, a')z\alpha(a'))$, where $\pi x g \pi x^{-1} = ha'$. Therefore we must verify that

$$c(\pi x g, \pi x^{-1})c(\pi x, g)c(h, \pi a)\alpha(a) = c(h, a')\alpha(a'),$$

which, since $a' = a \cdot g^{-1}\pi x g \pi x^{-1}$, simplifies to

$$c(\pi x, g\pi x^{-1})c(\pi x, g) = c(g, g^{-1}\pi x g \pi x^{-1})\alpha(g^{-1}\pi x g \pi x^{-1}). \tag{4.5}$$

In order to compute $\alpha(g^{-1}\pi x g \pi x^{-1})$ in (4.5), let $F : \Sigma \times [0,1] \to G$ be a homotopy such that $F_0 = x$ and $F_1 = e$. (Such a homotopy exists, since G is 2-connected.) Let $H : \Sigma \times [0,1] \to G'$ be defined by $H(p,t) = g(p)^{-1}\pi F(p,t) g(p) \pi F(p,t)^{-1}$, and argue as in the proof of Lemma 4.1 that

$$\alpha(g^{-1}\pi x g \pi x^{-1}) = \exp \frac{-i\pi}{6} \int_{\Sigma\times[0,1]} (g\pi F g^{-1}\pi F^{-1})^* \eta.$$

A direct calculation verifies that equation (4.5) holds.

The main strategy to verify (4.5) is to recognize $\rho = (g\pi F g^{-1}\pi F^{-1})^*\eta$ as a coboundary $\rho = d\tau$ and use Stokes' Theorem, so that

$$\int_{\Sigma\times[0,1]} \rho = \int_{\partial\Sigma\times[0,1]} \tau + \int_{\Sigma\times 0} \tau + \int_{\Sigma\times 1} \tau,$$

where

$$\tfrac{1}{6}\tau = B((\pi F)^*\theta^L, (g\pi F^{-1})^*\theta^R) + B((\pi F)^*\theta^L, g^*\theta^R) - B(g^*\theta^L, (g^{-1}\pi F g \pi F^{-1})^*\theta^R)$$

The first term does not contribute because $g|_{\partial\Sigma} = e$ and the third term above does not contribute because $F_1 = e$. □

The pre-quantum line bundle. As mentioned in the introduction, the construction of the pre-quantum line bundle over $\mathcal{M}'(\Sigma)$ appears in [1]. Nevertheless, the main steps in the construction are summarized next, focussing on the obstruction related to central extensions of the gauge group.

The pre-quantum line bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$ is obtained through a reduction procedure. Recall that $\widehat{\mathrm{Map}}(\Sigma, G')$ acts on the trivial bundle $\mathcal{A}(\Sigma)\times\mathbb{C}$ by

$$(g,w)\cdot(A,a) = (g\cdot A, \exp(-i\pi \int_\Sigma lB(g^*\theta^L, A))wa)$$

The 1-form $\alpha \mapsto \frac{1}{2}\int_\Sigma lB(A,\alpha)$ on $\mathcal{A}(\Sigma)$ defines an invariant connection, whose curvature can be verified to be $\omega_{\mathcal{A}}$.

By Corollary 4.3, when l is a multiple of $l_b(G')$ (see Definition 2.1), the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G') \subset \widehat{\mathrm{Map}}(\Sigma, G')$ splits, and we may define the pre-quantum line bundle over $\mathcal{M}'(\Sigma)$ by

$$L'(\Sigma) = (\mathcal{A}_{\mathrm{flat}}(\Sigma)\times\mathbb{C})/\mathrm{Map}_\partial(\Sigma, G').$$

As in the proof of Corollary 4.3, each choice of splitting of the central extension (4.4) induces a splitting of the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ over $\mathrm{Map}_\partial(\Sigma, G')$ used in the above construction. Since any two sections of the central extension (4.4) differ by a character $Z\times Z \to U(1)$, it is not hard to see that the set of pre-quantum line bundles are therefore in one-to-one correspondence with group of characters $\mathrm{Hom}(Z\times Z, U(1))$ (*cf.* Theorem 4.1(b) in [1]).

Finally, note that since the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$, where $\mathrm{Hol}(\mu)\in Z$, are the connected components of the moduli space $M_{G'}(\hat{\Sigma})$ of flat G'-bundles over the closed surface $\hat{\Sigma}$ (see the end of Section 3), the pre-quantum line bundle $L'(\Sigma)$ descends to a pre-quantization of $M_{G'}(\hat{\Sigma})$.

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Department of Mathematics and Statistics, McMaster University, Hamilton, ON
E-mail address: krepskid@math.mcmaster.ca